GENNAIO

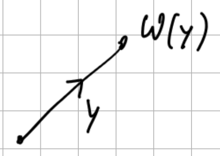
RECALL: graphs of groups

a graph of groups (Y, G) consists of a CONNECTED graph Y and

- vertex groups G_p, for $p \in Y^0$
- edge groups G_y, for $y \in Y^1$ $\quad (G_y = G_{\bar{y}})$

and INJECTIVE maps $G_y \longrightarrow G_{w(y)}$

(notation: $a \mapsto a^y$)

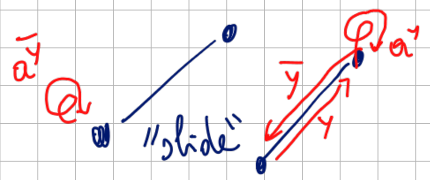


Today: define the FUNDAMENTAL GROUP of a graph of group and establish its basic properties.

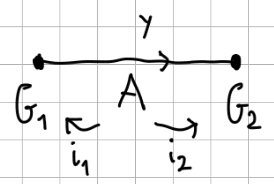
INTERMEDIATE STEP:

<u>Def</u> $$F(Y,G) = \Big(\underset{p \in \underbrace{Y^0}_{\text{vertices}}}{\ast} G_p \Big) \ast F_{\underbrace{Y^1}_{\text{edges}}} \Big/ \langle\!\langle \bar{y} = y^{-1},\ y\, a^y\, \bar{y} = a^{\bar{y}} \rangle\!\rangle$$

for $y \in Y^1$, $a \in G_y$.

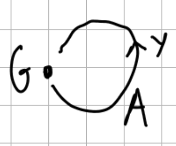


<u>Example</u>



$$\Rightarrow \quad F(Y,G) = \frac{G_1 \ast G_2 \ast \mathbb{Z}^{=\langle y \rangle}}{\langle\!\langle y\, i_2(a)\, y^{-1} = i_1(a) \rangle\!\rangle_{a \in A}}$$

<u>Example</u>



$$\Rightarrow \quad F(Y,G) = \frac{G \ast \mathbb{Z}^{=\langle y \rangle}}{\langle\!\langle y\, i_2(a)\, y^{-1} = i_1(a) \rangle\!\rangle_{a \in A}}$$

DEFINITIONS OF π_1

- relative to a spanning tree $T \subseteq Y$:

$$\pi_1(Y, G; T) = F(Y,G) \big/ \langle\!\langle y, \ y \in T^1 \rangle\!\rangle$$

ex : $\bullet\!\longrightarrow\!\bullet$ $\Rightarrow$ amalgamated product

$Y = T$ $\Rightarrow$ colimit we have seen in previous lectures

$G \bullet\!\bigcirc A$ $\Rightarrow$ $T = \bullet$, $\pi_1 = F(Y, G)$

It is called an HNN extension (of G)

- relative to a vertex $P \in Y^0$:

Def If $C = \underset{P_0}{\bullet} \xrightarrow{y_1} \underset{P_1}{\bullet} - - - \xrightarrow{y_n} \underset{P_n}{\bullet}$ is a path in Y,

a word of type c is a sequence

$$\mu = (\mu_0, \dots, \mu_n) \qquad \mu_i \in G_{P_i} .$$

It has an associated element

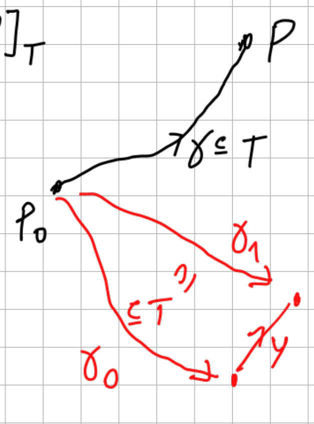
$$\mu_0\, y_1\, \mu_1\, y_2 \dots y_n\, \mu_n \in F(Y, G).$$

Def $\pi_1(Y, G; P_0) = \{$ elements associated to words of types with extremities $= P_0 \} < F(Y, G)$.

Prop The projection $F(Y, G) \to \pi_1(Y, G; T)$ restricts to an isomorphism $\pi_1(Y, G; P_0) \cong \pi_1(Y, G; T)$.

Pf : define the inverse $\pi_1(Y, G; T) \to \pi_1(Y, G; P_0)$.

$$g \in G_P \longmapsto \gamma g \gamma^{-1} , \quad \gamma = [P_0, P]_T$$



$$y \in Y^1 \longmapsto \gamma_0\, y\, \gamma_1^{-1}$$

This defines hom.

$$(\ast G_p) \ast F_{Y^1} \longrightarrow \pi_1(Y, G; P_0)$$

<u>obs</u> $y \in T^1 \Rightarrow \gamma_0\, y\, \gamma_1^{-1} = 1$.

So, the hom. passes to

$$\pi_1(Y, G; T) \longrightarrow \pi_1(Y, G; P_0).$$

Exercise: it is the inverse of the projection. □

A TECHNICAL PROPOSITION

<u>Def</u> The <u>trivial</u> word: type $c = \overset{\bullet}{P_0}$ $(n=0)$,

$$\mu_0 = 1_{G_{P_0}}$$

<u>Def</u> Let $c = y_1 \dots y_n$ be a path in Y.
A word $\mu = (\mu_0, \dots, \mu_n)$ of type c is REDUCED if:

whenever $y_{i+1} = \overline{y_i}$ (backtrack),
it holds $\mu_i \notin (G_{y_i})^{y_i}$

NOTE: if $\mu_i = (a_i)^{y_i}$, then

$$\dots\ y_i\ (a_i)^{y_i}\ \overline{y_i}\ \dots = \dots\ (a_i)^{\overline{y_i}} \dots \in F(Y, G).$$

<u>Prop</u> The element $\in F(G, Y)$ associated to a <u>nontrivial reduced word</u> is $\neq 1 \in F(G, Y)$.

<u>Corollaries</u>:

- $\forall p \in Y^0$, $G_p \to F(Y, G)$ is injective
- (c, μ) reduced of length $n \geq 1$ $\rightsquigarrow$ element $\notin G_{P_0}$ $\quad P_0 = o(y_1)$

• If c is a closed path, the element associated to a nontrivial reduced word of type c "survives" in $\pi_1(Y, G; T)$

(where T is any maximal subtree) $\left[\begin{array}{l}\text{In particular,} \\ G_P \hookrightarrow \pi_1(Y, G; T)\end{array}\right]$

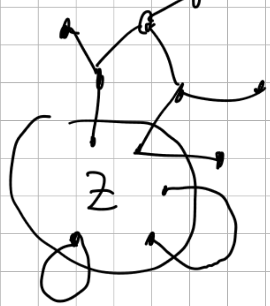
Pf: the associated element belongs to $\pi_1(Y, G; P_0)$ and is nontrivial. The projection $\to \pi_1(Y, G; T)$ is an iso. with $\pi_1(Y, G; P_0)$. □

PROOF OF THE TECHNICAL PROPOSITION

Fact : collapsing connected subgraphs.

Let $Z \subset Y$ be a connected subgraph. Suppose Z satisfies the proposition. Consider the "quotient" graph of groups

$$(Y/Z \quad , \quad G')$$



$$G'_{Z} = F(Z, G|_Z)$$

Vertex in Y/Z

Then, there is a natural $F(Y, G) \to F(Y/Z, G')$, which is an isomorphism (exercise).

We also have a function between words:

$$(c, \mu) \text{ in } (Y, G) \quad \rightsquigarrow \quad (c', \mu') \text{ in } (Y/Z, G')$$

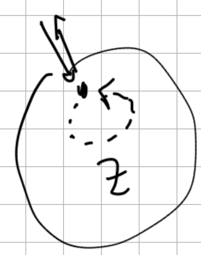
"aggregate subpaths $\subseteq Z$"

Lemma Suppose Z satisfies the technical proposition.
(c, μ) reduced nontrivial $\Rightarrow$ (c', μ') reduced nontrivial.

Pf: $n' = 0$ outside Z $\Rightarrow$ OK ✓

Z $\Rightarrow$ OK, by corollary, applied to Z. ✓

$n' \geq 1$: vertices outside Z are OK.

If  , OK because of the third corollary, applied to Z. $\square$

Consequence: if both Z and Z/Y satisfy the technical prop., then also Y does.

Strategy: prove the prop. for "easy" graphs and then treat more complicated ones using subgraphs and quotients.

<u>CASE: SEGMENT</u> $\qquad Y = P \overset{}{\underset{y}{\bullet\!\!\longrightarrow\!\!\bullet}} Q$

Reduced nontrivial word gives:

$$\begin{array}{cccccccc} g_0 & y & g_1 & \bar{y} & g_2 & \cdots & & g_n \\ \cap & & \cap & & \cap & & & \\ G_P & & G_Q \setminus G_y^y & & G_P \setminus G_y^{\bar{y}} & & & \end{array}$$

or with G_P and G_Q swapped (and $y \leftrightarrow \bar{y}$).

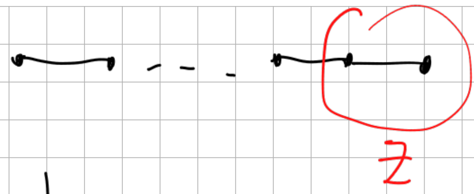
The image in $\pi_1(Y, G; T) = G_P \underset{G_y}{*} G_Q$ is

$g_0 \cdots g_n$, which easily gives a nontrivial reduced word for the amalgamated product, <u>unless</u> we have

$g_0\, y\, g_1$ or $g_0\, \bar{y}\, g_1$ with $g_0, g_1 \in$ images of G_y, g_0 the inverse of g_1. In this case, $g_0\, y\, g_1 = y \in F(Y, G)$,

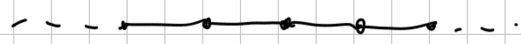
or $g_0\, \bar{y}\, g_1 = \bar{y} \in F(Y, G)$, which nonetheless is nontrivial: relations in $F(Y, G)$ preserve the $(\#\text{ of } y - \#\text{ of } \bar{y})$.

CASE: FINITE LINE

$Y =$ 

By induction on the length!

CASE: BI-INFINITE LINE

$Y =$ 

A nontrivial reduced word "lives" in a finite subgraph.
Any proof of triviality of the associated element would use a finite number of relations, all involving "letters" living in a finite subgraph.
Therefore, the conclusion follows from the "finite line" case.

CASE: LOOP

$Y =$ 

$$F(Y,G) = G_p * \underset{\langle y \rangle}{\underset{\cong}{\mathbb{Z}}} \,/\, \langle\langle y\, a^y\, y^{-1} = a^{\bar{y}} \text{ for } a \in G_y \rangle\rangle$$

If c has strictly more "y" than "$\bar{y}$", or viceversa, then the associated element is nontrivial, because it survives under the natural map

$$F(Y,G) \longrightarrow \mathbb{Z}$$

induced from $G_p * \mathbb{Z} \longrightarrow \mathbb{Z}$.
From an old exercise, we know that

$$\mathrm{Ker}\,(G_p * \mathbb{Z} \to \mathbb{Z}) = \mathop{\ast}_{n \in \mathbb{Z}} (y^n G_p y^{-n}).$$

It follows:

$$\mathrm{Ker}\,(F(Y,G) \to \mathbb{Z}) = \frac{\mathop{\ast}_{n\in\mathbb{Z}} (y^n G_p y^{-n})}{\langle\langle y^n a^y y^{-n} = y^{n-1} a^{\bar{y}} y^{-(n-1)} \rangle\rangle_{\substack{a \in G_y \\ n \in \mathbb{Z}}}}$$

On the other hand, consider the infinite line of groups (L, H):

$$L = \quad \cdots \bullet \xrightarrow[H]{} \overset{H_0}{\bullet} \xrightarrow[H]{} \overset{H_1}{\bullet} \xrightarrow[H]{} \bullet \cdots \qquad \begin{aligned} H &= G_y \\ H_n &= y^n G_p y^{-n} \end{aligned}$$

(each H_n is a subgroup of $G_p * \mathbb{Z}$)

with inclusion maps:

$$H_{n-1} \ni y^{n-1} a^{\bar{y}} y^{-(n-1)} \longleftarrow a \in H \longrightarrow y^n a^y y^{-n} \in H_n$$

Then:

$$F(L,H) = \mathop{\ast}_{n\in\mathbb{Z}} y^n G_p y^{-n} \Big/ \langle\langle\, y^{n-1} a^{\bar{y}} y^{-(n-1)} = y^n a^y y^{-n} \,\rangle\rangle$$

which is exactly the $\mathrm{Ker}\,(F(Y,G) \to \mathbb{Z})$ computed above.
If (c,μ) is a nontrivial reduced word in (Y,G), with $\# y = \# \bar{y}$ (i.e., the associated elements $\in \mathrm{Ker}(F(Y,G) \to \mathbb{Z})$) then, we have (naturally) a closed nontrivial reduced word in (L,H):

$$\begin{tikzcd}
(c,\mu) \arrow[r, dashed] \arrow[d, dotted] \arrow[dr, phantom, "\circlearrowright"] & (c',\mu') \text{ in } (L,H) \arrow[d, dotted] \\
\mathrm{Ker}\,(F(Y,G) \to \mathbb{Z}) \arrow[r, phantom, "="] & F(L,H)
\end{tikzcd}$$

Therefore, we conclude using the "bi-infinite line" case.

CASE: FINITE GRAPH

By induction on # edges: collapse $\bullet\!-\!\bullet$ or $\circlearrowright$ at each step.

GENERAL CASE By "finiteness argument", exactly the same as for the infinite line. $\square$