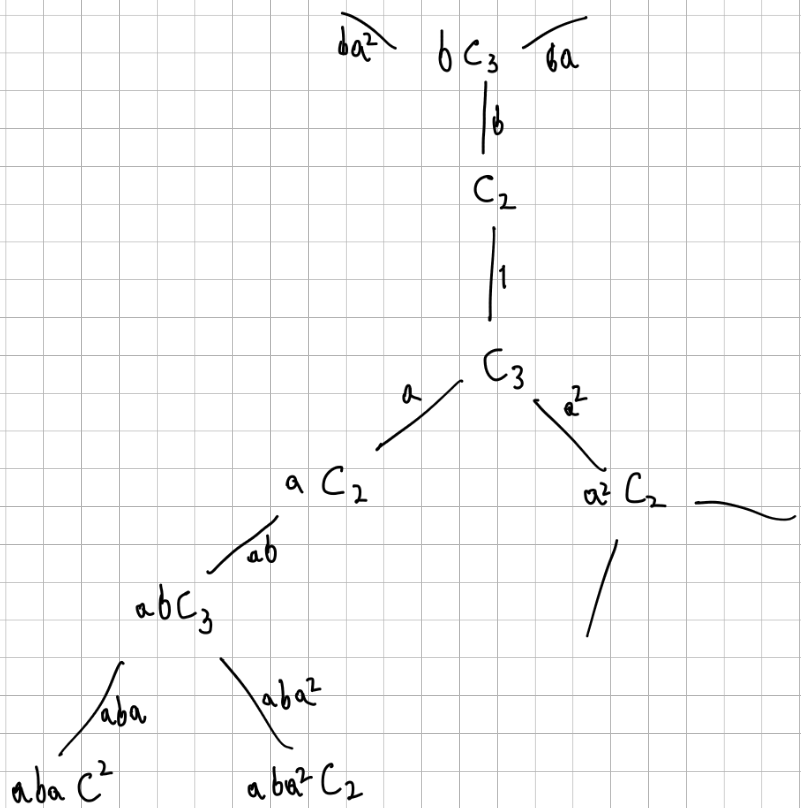
Example $G = C_3 * C_2$, $\quad (C_n := \mathbb{Z}/n\mathbb{Z})$

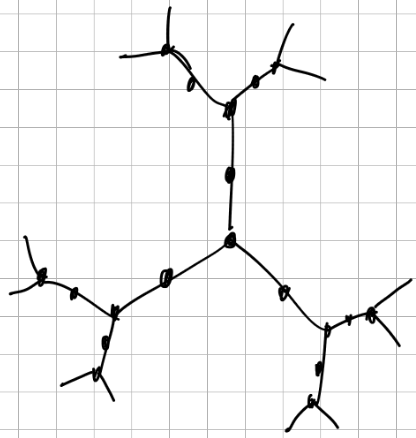
What is the associated tree X? $\quad C_3 = \langle a \mid a^3 \rangle \quad C_2 = \langle b \mid b^2 \rangle$

$gC_3 \in X^0 \qquad st(gC_3) = \{h\{1\} \mid h\{1\} = g\{1\}\}$

$\qquad\qquad\qquad\qquad\qquad = gC_3$

$gC_2 \in X^1 \qquad st(gC_2) = gC_2.$





a acts as an elliptic elements "rotating" around C_3.

b -- -- -- C_2.

ab? Hyperbolic, with axis passing through

$\ldots ba^2C_2,\ bC_3,\ C_2,\ C_3,\ aC_2,\ abC_3,\ abaC_2,\ ababC_3 \ldots$

translation length: 2.

EX G group. $f: \underset{\langle t \rangle}{\overset{\shortparallel}{\mathbb{Z}}} * G \to \mathbb{Z}$ induced by id and trivial.

Then, $\operatorname{Ker} f = \mathop{\ast}\limits_{n \in \mathbb{Z}} t^n G t^{-n}$.

$SL_2(\mathbb{Z})$

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{Z},\quad ad-bc=1 \right\}.$$

Who knows a bit of hyperbolic or complex geometry, knows how $SL_2(\mathbb{Z}) < SL_2(\mathbb{R})$ acts on $\{z \in \mathbb{C} \mid \operatorname{im} z > 0\}$.
One can find a fundamental domain and see that its translates are arranged in a tree-like fashion, from which one finds an action of $SL_2(\mathbb{Z})$ on a tree.
Let us see another perspective (to obtain the same thing).

FAREY BINARY SEARCH

$L_0, R_0 \in \mathbb{R}$ $\qquad L_0 < R_0$

$U \in (L_0, R_0)$ $\qquad$ "unknown".

Standard binary search: $\quad M_0 := \frac{L_0+R_0}{2}$.

Ask if $U = M_0$ or $U \in (L_0, M_0)$ or $U \in (M_0, R_0)$;

iterate with $(L_1, R_1) = (L_0, M_0)$ or (M_0, R_0) ...

FAREY binary search:

start with <u>rational</u> range $L_0 = \frac{a}{b}$, $R_0 = \frac{c}{d} \in \mathbb{Q}$, reduced fractions with $b, d > 0$.

To split the searching range, pick $M_0 = \dfrac{a+c}{b+d} \in \mathbb{Q}$ "MEDIANT"
and proceed analogously.

NOTE: $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

<u>Exercise</u>: For every $U \in \mathbb{Q} \cap (L_0, R_0)$
the procedure eventually arrives at $M_K = U$ for a certain $K \in \mathbb{N}$.

[Hint: if $0 < \frac{a}{b} < \frac{x}{y} < \frac{c}{d}$ (reduced fractions), then $(bc-ad)(x+y) \geq a+b+c+d$.]

For our purposes, it is convenient to extend to $\mathbb{P}^1\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$.

<u>Def</u> Given $x, y \in \mathbb{P}^1\mathbb{Q}$ distinct, they have TWO MEDIANTS:

write $x = \frac{a}{b}$, $y = \frac{c}{d}$ as <u>reduced</u> integer fractions. Then, ↘ i.e.: $a, b \in \mathbb{Z}$, $\gcd(a,b) = 1$.

the mediants are: $\frac{a+c}{b+d}$, $\frac{a-c}{b-d}$.

<u>Def</u> Given $x, y \in \mathbb{P}^1\mathbb{Q}$, write $\Delta(x,y) = \left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right|$ if $x = \frac{a}{b}$, $y = \frac{c}{d}$ as reduced integer fractions.

<u>Lemma</u> If $\Delta(x,y) = 1$ and z is a mediant of x and y, then: $\Delta(x,z) = \Delta(y,z) = 1$; x is a mediant of y and z
y is a mediant of x and z.

Pf: $x = \frac{a}{b}$, $y = \frac{c}{d}$ reduced fractions. $z = \frac{a+c}{b+d}$.

$$\left|\det\begin{pmatrix} a & a+c \\ b & b+d \end{pmatrix}\right| = \left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = \Delta(x,y) = 1.$$

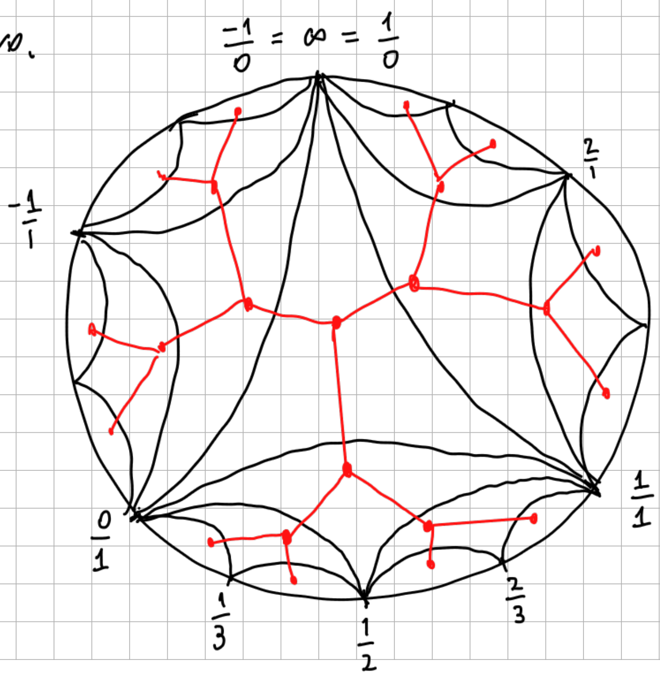
In particular $a+c$, $b+d$ are coprime, so $\Delta(x,z) = 1$

and $\frac{a}{b} = \frac{(-c) + (a+c)}{(-d) + (b+d)}$ $\square$

Procedure: Start from the triple $\{0, 1, \infty\} \subset \mathbb{P}^1\mathbb{Q}$; notice that $\Delta(0, \infty) = 1$ and that 1 is a mediant of 0 and ∞; at each step, replace one of the elements by <u>the other</u> mediant of the remaining two.

obs: at any instant we have $\{x, y, z\}$ s.t. $\underbrace{1 = \Delta(x,y) = \Delta(y,z) = \Delta(x,z)}_{(*)}$ and each member is a mediant of the other two.

We obtain a <u>tree</u> of possible outcomes!



Def $SL_2(\mathbb{Z}) \curvearrowright \mathbb{P}^1\mathbb{Q}$ as $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \frac{p}{q} = \frac{ap+cq}{bp+dq}$.

Obs: If $p, q \in \mathbb{Z}$, then $\gcd(p,q) = \gcd(ap+cq, bp+dq)$

obs Let $M \in SL_2(\mathbb{Z})$, $x, y \in \mathbb{P}^1\mathbb{Q}$. Then, $\Delta(x,y) = \Delta(Mx, My)$.

Lemma If $x, y \in \mathbb{P}^1\mathbb{Q}$ and $\Delta(x,y) = 1$, then $\exists M \in SL_2(\mathbb{Z})$ s.t. $Mx = \infty$, $My = 0$.

Pf: $x = \frac{a}{b}$, $y = \frac{c}{d}$, $\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = 1$.

Take $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1}$. □

Lemma If $\Delta(x,y) = 1$, then there are exactly 2 points $z \in \mathbb{P}^1\mathbb{Q}$ such that $\Delta(x,z) = \Delta(y,z) = 1$: the medients of x and y.

Pf: We know that the medients satisfy the condition.

Enough to show: there are only two points satisfying the cond.

If $x = \infty = \frac{1}{0}$ and $y = 0 = \frac{1}{0}$, the conditions are:

$\det\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} = \pm 1$, $\det\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = \pm 1$, which imply

$\frac{a}{b} \in \{\pm 1\}$.

For generic x, y: let $M \in SL_2(\mathbb{Z})$ be s.t. $Mx = \infty$, $My = 0$.

Then, z satisfies $\Delta(x,z) = \Delta(y,z) = 1$, which implies

$$\Delta(Mx, Mz) = \Delta(Mx, Mz) = 1,$$

and there are only 2 possibilities for Mz. □

Cor Let $x, y \in \mathbb{P}^1\mathbb{Q}$ with $\Delta(x,y) = 1$. Let $M \in SL_2(\mathbb{Z})$, $z \in \mathbb{P}^1\mathbb{Q}$.

z is a mediant of x and y $\Leftrightarrow$ $\Delta(x,z) = \Delta(y,z) = 1$

$\Updownarrow$

Mz - - - - - Mx and My. $\Leftrightarrow$ $\Delta(Mx, Mz) = \Delta(My, Mz) = 1$

Lemma Let $x, y, z \in \mathbb{P}^1\mathbb{Q}$ satisfy $(*)$. Then, $\{x,y,z\}$ is reachable starting from $\{0, 1, \infty\}$.

Pf: First, suppose $x = \infty$. Then, $(*)$ implies that $\{\infty, y, z\} = \{\infty, n, n+1\}$ for some $n \in \mathbb{Z}$.

They are all reachable:

$$\{\infty, n-1, n\} \longleftrightarrow \{\infty, n, n+1\}$$

because the mediants of ∞ and $n \in \mathbb{Z}$ are $n-1$ and $n+1$.

Let $\{x,y,z\}$ be a triple satisfying $(*)$.
We know from the exercise that $\exists$ reachable triple containing, e.g., x. Say, $\{x, y', z'\}$.
Let $M \in SL_2(\mathbb{Z})$ s.t. $Mx = \infty$.
Then, $\{Mx, My', Mz'\}$ and $\{Mx, My, Mz\}$ satisfy $(*)$ and contain ∞, so they are of the form

$$\{\infty, n', n'+1\} \qquad \{\infty, n, n+1\}$$

these are "connected" by a sequence of steps, so the same is true for $\{x,y,z\}$ and $\{x,y',z'\}$, by the previous Corollary. $\square$

Conclusion: we have a tree (the FAREY TREE) X with:

$$X^0 = \{\{x,y,z\} \subseteq \mathbb{P}^1\mathbb{Q} \text{ satisfying } (*)\}$$

$$X^1 = \{(\{x,y,z\}, z) : \{x,y,z\} \in X^0\}$$

$$\begin{array}{ccc} & (\{x,y,z\}, z) & \\ \swarrow \alpha & & \searrow \omega \\ \{x,y,z\} & & \{x, y, z' \text{ the other mediant}\} \end{array}$$

$SL_2(\mathbb{Z}) \curvearrowright X$ from the action on $\mathbb{P}^1\mathbb{Q}$.

Lemma: $SL_2(\mathbb{Z}) \curvearrowright X^1$ is transitive.

Pf: Let $(\{x,y,z\}, z) \in X^1$ we can write:

$$x = \frac{a}{b}, \quad y = \frac{c}{d} \qquad z = \frac{a+c}{b+d}.$$

Up to swapping x, y, we have $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL_2(\mathbb{Z})$.
Then, $M(\{0,\infty,1\}, 1) = (\{x,y,z\}, z)$. $\square$

Obs In particular, $SL_2(\mathbb{Z}) \curvearrowright X^0$ is transitive, and there are inversions!

$Z :=$ baricentric subdivision of X.
Then, $SL_2(\mathbb{Z}) \curvearrowright Z$ with segment quotient!
Fundamental domain:

$$\{0,\infty,1\} \xrightarrow{\;y\;} (\{0,\infty,1\},1) \sim (\{0,\infty,-1\},-1)$$

Stabilizers?

- Of $\{0,\infty,1\}$: $M \in SL_2(\mathbb{Z})$ must permute $0, \infty, 1$.

$$\left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \pm\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \; \pm\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right\} \cong \mathbb{Z}/6\mathbb{Z}$$

$\infty \to 0 \to 1$ $\qquad$ $\infty \leftarrow 0 \leftarrow 1$

- Of $(\{0,\infty,1\},1) \sim (\{0,\infty,-1\},-1)$

$$\left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \pm\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \cong \mathbb{Z}/4\mathbb{Z}$$

$0 \leftrightarrow \infty \quad 1 \leftrightarrow -1$

- Stabilizer of y: the intersection: $\left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \cong \mathbb{Z}/2\mathbb{Z}$.

$$\Rightarrow SL_2(\mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z} \underset{\mathbb{Z}/2\mathbb{Z}}{*} \mathbb{Z}/4\mathbb{Z}$$

presentation: $\langle x, y \mid x^6, y^4, x^3 = y^2 \rangle$.

$$x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$