PRESENTATIONS OF GROUPS

S set, R a subset of F_S.

Def $\langle S \mid R \rangle := F_S / \langle\langle R \rangle\rangle$

"NORMAL CLOSURE"

where $\langle\langle R \rangle\rangle$ is the smallest normal subgroup containing R.

Def A presentation of a group G is a package containing:

- a set S with
- a function $S \to G$;
- a subset $R \subseteq F_S$ (or a set of equations "$w_1 = w_2$", $w_i \in F_S$, that we interpret as $w_1 w_2^{-1} \in F_S$)

such that $F_S \to G$ is surjective with kernel $\langle\langle R \rangle\rangle$ (hence, it induces $\langle S \mid R \rangle \xrightarrow{\cong} G$).

"Let $\langle S, R \rangle$ be a presentation of G" means that we have fixed a map $S \to G$ s.t. we are in the situation above.

Example
- $\langle x, y \mid xy = yx \rangle$ is a presentation of $\mathbb{Z}^2$ if we send, e.g., $x \to (1,0)$ and $y \to (0,1)$.
- $\langle x \mid x^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$

It is both a way to describe a group that we already have, and a way to construct new groups.

AMALGAMATED PRODUCTS

<u>Def</u> Let G be a group, $A < G$ a subgroup, $\{G_i\}_{i\in I}$ a nonempty family of subgroups of G, with $A < G_i$ $\forall i \in I$.
For $n \in \mathbb{N}$, let $i_1, \dots, i_n \in I$ be such that $i_{m+1} \neq i_m$ for $m \in \{1, \dots, n-1\}$. A REDUCED WORD of type $\underline{i} = (i_1, \dots, i_n)$ is a sequence $a\, g_1 \dots g_n$ with: $a \in A$, $g_m \in G_{i_m} \setminus A$.
We say that $\underline{n}$ is the <u>length</u> of the reduced word.
We say that G is the amalgamated product of the G_i's along A if $\forall g \in G$ $\exists!\, n \in \mathbb{N}$, $\underline{i} \in I^n$ as above such that $g = a\, g_1 \dots g_n$ for some reduced word of type $\underline{i}$, and any other such reduced word is of the form

$$(a\, a_1^{-1})(a_1 g_1 a_2^{-1}) \dots (a_n^{-1} g_n).$$

"CONTROLLED NON UNIQUENESS"

We write $G = \mathop{\ast}_{A}^{i\in I} G_i$.

<u>NOTE</u> If $A = \{1\}$, G is called a FREE PRODUCT of the G_i's, and write $G = \mathop{\ast}^{i\in I} G_i$.
In this case, $\forall g \in G$ the expression is unique.

<u>NOTE</u> If $|I| = 2$, we write $G = G_1 \ast_A G_2$.

<u>Rem</u>
- Any $g \in G$ is either $a \in A$ $(n=0)$, or $g_1 \dots g_n$ $(n \geq 1)$ reduced (with $a = 1$, $n \geq 1$) $\notin A$
- $i_1 \neq i_2 \Rightarrow G_{i_1} \cap G_{i_2} = A$
- If $A = \{1\}$, $G_i = \langle s_i \rangle$ infinite cyclic, then G is free with basis $\{s_i : i \in I\}$.

Any (non reduced) $g = g_1 \dots g_m$ can be written in reduced form by "aggregating" consecutive elements $\in$ same G_i.

Prop G group, $A < G$, $\{G_i\}_{i\in I}$ nonempty family of subgroups.
Suppose that G is generated by $\bigcup_{i\in I} G_i$.
Then: G is the amalgamated product of the G_i's
$\Leftrightarrow$ the only reduced word for 1_G is 1_G (length 0).

Proof: $\Rightarrow$: by definition.

$\Leftarrow$ Suppose $a\, g_1 \dots g_m = b\, h_1 \dots h_k$ are reduced words of types $\underline{i} = (i_1, \dots, i_m)$, $\underline{j} = (j_1, \dots, j_k)$

(We have to prove that: $m = k$, $\underline{i} = \underline{j}$, and that $\exists\, a_1, \dots, a_m$ as in the def. of amalg. prod.)

- if $m = 0$ or $k = 0$: ok

Now: $k \geq 1$. Proceed by induction on m. $m = 0$: done. $m \geq 1$:

$a\, g_1 \dots g_m = b\, h_1 \dots h_k$

$a\, g_1 \dots g_{m-1} = \underbrace{b\, h_1 \dots h_k\, g_m^{-1}}_{\text{cannot be reduced!}}$ (otherwise: by inductive hyp, $i_{m-1} = i_m \dots$)

$\Rightarrow i_m = j_k$.

$a\, g_1 \dots g_{m-1} = b\, h_1 \dots h_{k-1} (h_k g_m^{-1})$

If $h_k g_m^{-1} \notin A$, these are reduced words. Apply induction:

$m-1 = k$, $b = a\, a_1^{-1}$, $h_1 = a_1 g_1 a_2^{-1} \dots$

$h_k g_m^{-1} = a_k\, g_{m-1}$

$a_k = (h_k g_m^{-1})\, g_{m-1}^{-1}$ reduced, contradiction. So, actually:

$h_k g_m^{-1} \in A$. $a\, g_1 \dots g_{m-1} = b\, h_1 \dots (h_{k-1} h_k g_m^{-1})$ are reduced.

$\rightarrow m-1 = k-1$, $\underline{i} = \underline{j}$, $b = a\, a_1^{-1}$, $h_1 = a_1 g_1 a_2^{-1}$

$\dots\ h_{k-1} (h_k g_m^{-1}) = a_{k-1}\, g_{m-1}$

i.e.: $h_{k-1} = a_{k-1}\, g_{m-1} (h_k g_m^{-1})^{-1}$, and obviously:

$h_k = (h_k g_m^{-1})\, g_m$ $\square$

Universal property : $\forall$ group H, $\forall \{f_i : G_i \to H\}$ fam. of homomorphisms agreeing on A, $\exists! \ f : G \to H$ extending all the f_i's.

$$\left(f(g_1 \cdots g_m) = f_{i_1}(g_1) \cdots f_{i_m}(g_m) \quad \text{if } g_m \in G_{i_m} \right)$$

Construction Given:

- a group A
- a nonempty family of groups $\{G_i\}$
- injective homomorphisms $J_i : A \to G_i$,

There is a standard construction of a group G and injective maps $\varphi : A \to G$, $\varphi_i : G_i \to G$ such that $G = \underset{\varphi(A)}{*} \varphi_i(G_i)$.

NOTE: we simply write $G = \underset{A}{*} G_i$.

obs : (free product of free groups) $\underset{i \in I}{*} F_{S_i} = F_{\sqcup S_i}$.

ex : $* \, \mathbb{Z}$ is a free group of rank $|I|$.

Prop $* \langle S_i \mid R_i \rangle = \langle \sqcup S_i \mid \sqcup R_i \rangle$

Pf: $\langle \sqcup S_i \mid \sqcup R_i \rangle$ satisfies the univ. property of $* \langle S_i \mid R_i \rangle$.

Example: $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle x, y \mid x^2 = y^2 = 1 \rangle$

Prop Let A, $\{G_i\}$, $\{J_i : A \to G_i\}$ as in the construction. Then, $\underset{A}{*} G_i$ is canonically isomorphic to $* G_i / \langle\langle J_{i_1}(a)^{-1} J_{i_2}(a) \rangle\rangle$

$i_1, i_2 \in I$, $a \in A$.

Pf: That quotient satisfies the universal property of the amalgamated product.

FUNDAMENTAL DOMAINS FOR $G \curvearrowright X$

<u>Def</u> Let $G \curvearrowright X$ be an action on a graph, without inversions.
A fundamental domain for $G \curvearrowright X$ is a subgraph $D \subseteq X$ such that $D \to G \backslash X$ is an isomorphism.

<u>Prop</u> Suppose X is a tree. Then:

$G \curvearrowright X$ admits fundamental domains $\Leftrightarrow$ $G \backslash X$ is a tree.

Pf: $X \to G \backslash X$ is locally surjective.

- If $G \backslash X$ is a tree, we can lift it and find D.
- If D is a fundamental domain, then D is a subgraph of X which is connected (because $G \backslash X$ is connected), so it is a subtree, and $G \backslash X \cong D$ is a tree. $\square$

STABILIZERS

<u>Notation</u>: If $G \curvearrowright X$, and $P \in X^0$, $G_P := \{ g \in G \mid gP = P \}$

$y \in X^1$, $G_y := \{ g \in G \mid gy = y \}$.

NOTE: if $P = \alpha(y)$ or $\omega(y)$, then $G_y < G_P$.

ACTIONS WITH "SEGMENT" QUOTIENT

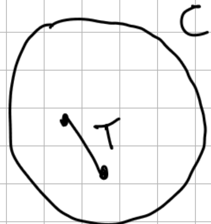
<u>Theorem</u> Let $G \curvearrowright X$ be an action without inversions,

$T = P \xrightarrow{y} Q$ a "segment" subgraph of X.

Suppose that T is a fundamental domain for $G \curvearrowright X$.
Then, the following properties are equivalent:

1) X is a tree
2) $G = G_P \ast_{G_y} G_Q$.

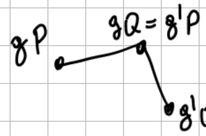
Proof: <u>STEP 1</u> X is connected $\Leftrightarrow$ G is generated by $G_P \cup G_Q$.



C connected component containing T

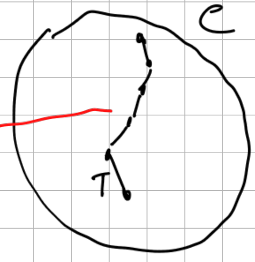
- Both G_P and G_Q preserve C.
Hence, $\langle G_P \cup G_Q \rangle$ preserve C.
$\langle G_P \cup G_Q \rangle T \subseteq C$.

- If $g, g' \in G$ are such that $gT \cap g'T \neq \emptyset$, then either
  $gP = g'P$ or $gQ = g'Q$.
  $g^{-1}g' \in G_P$ or $\in G_Q$
  $g^{-1}g' \in \langle G_P \cup G_Q \rangle$.



No! P, Q are in different orbits.

This implies that $\langle G_P \cup G_Q \rangle T = C$.

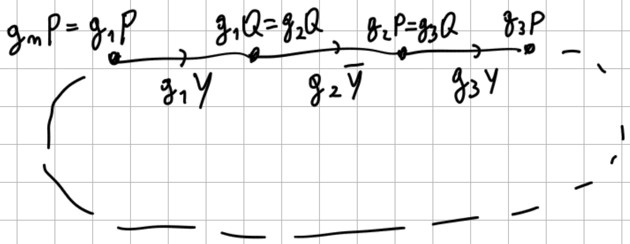


each edge is a G-translate of T

- If $G = \langle G_P \cup G_Q \rangle$, then $X = GT = C$, so X is connected.
- If X is connected, then $\langle G_P \cup G_Q \rangle T = C = X$, so
  $\forall g \in G \quad gT = g'T$ for some $g' \in \langle G_P \cup G_Q \rangle$
  $\Rightarrow g^{-1}g' \in \langle G_P \cup G_Q \rangle \Rightarrow g \in \langle G_P \cup G_Q \rangle$

<u>STEP 2</u> $(2) \Rightarrow (1)$.

By step 1, X is connected. Suppose it is <u>not</u> a tree.



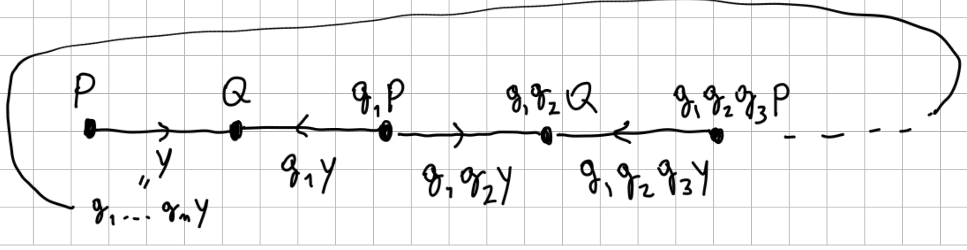
$g_1^{-1} g_2 \in G_Q \setminus G_y$
$g_2^{-1} g_3 \in G_P \setminus G_y$
$\vdots$
$g_m^{-1} g_1 \in G_P \setminus G_y$

But then $(g_1^{-1}g_2)(g_2^{-1}g_3)\cdots(g_m^{-1}g_1) = 1_G$, which implies that G is <u>not</u> the amalgamated product of G_P and G_Q.

<u>STEP 3</u>: $(1) \Rightarrow (2)$.

By step 1, $G = \langle G_P \cup G_Q \rangle$. If it is not the amalgamated product, then $1_G = g_1 \cdots g_m$ with $n \geq 1$,
$g_1 \in G_Q \setminus G_y$, $G_2 \in G_P \setminus G_y$, ... (or with P, Q swapped)

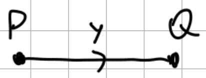
Then:



would be a closed reduced path of length ≥ 1 ↯.

□

<u>Theorem</u> Suppose $G = G_1 \underset{A}{*} G_2$. Then, $\exists$ tree X and an action $G \curvearrowright X$ with fundamental domain a segment

$T =$ 

such that $G_P = G_1$, $G_Q = G_2$, $G_y = A$.

Pf:

$$X^0 = G/G_1 \sqcup G/G_2$$
$$X^1 = G/A \sqcup \overline{G/A}$$

$$\alpha(gA) = gG_1$$
$$\omega(gA) = gG_2$$

these are forced!

$P = 1_G G_1$ $Q = 1_G G_2$

$y = 1_G A$

We have the action $G \curvearrowright X$ (multiplication).

T is a fundamental domain.

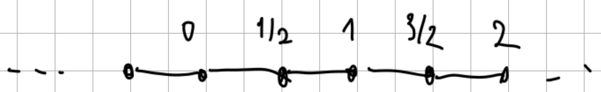
By the previous theorem, X is a tree. $\square$

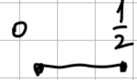
<u>Example</u> $G = \mathrm{Isom}(\mathbb{Z}) = \{x \mapsto ax+b \mid a \in \{\pm 1\}, b \in \mathbb{Z}\}$
$= \mathrm{Aut}(L)$

$L =$  $= \Gamma(\mathbb{Z}, \{1\})$.

L is a tree. But $G \curvearrowright L$ has inversions.

$T =$ baricentric subdivision of L.

$G \curvearrowright$ 

The quotient is a segment! Fund. domain: e.g., 

$$G_0 = \{x \mapsto x,\ x \mapsto -x\} \cong \mathbb{Z}/2\mathbb{Z}$$
$$G_{1/2} = \{x \mapsto x,\ x \mapsto 1-x\} \cong \mathbb{Z}/2\mathbb{Z}$$

The stabilizer of the edge is trivial: $\{x \mapsto x\}$.

$$\Rightarrow \quad G = G_0 * G_{1/2} = \langle a, b \mid a^2 = 1,\ b^2 = 1 \rangle$$