REVIEW OF FREE GROUPS

Let G be a group. A subset $S \subseteq G$ is a <u>basis</u> if

$\forall\ g \in G \quad \exists!\ k \in \mathbb{N}\ ,\quad s_1, \dots, s_k \in S\ ,\quad \varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$

such that $s_i = s_{i+1}$ implies $\varepsilon_i = \varepsilon_{i+1}$, and $g = 1_G\, s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k}$.

A group is free if it has a basis (unless $G = \{1_G\}$, the basis is not unique.)

<u>Universal property</u> :

If S is a basis of G,

$$\begin{tikzcd} S \arrow[r] \arrow[d] & H \text{ group} \\ G \arrow[ur, dashed, "\exists!\ \text{homomorphism}"'] & \end{tikzcd}$$

(easy proof)

Given a set S, there is a canonical construction of a free group F_S with basis S.

<u>Def</u> Let G be a group. $\operatorname{rank}(G)$ is the minimum cardinality of a subset $S \subseteq G$ of generators.

<u>Exercise</u> If G has a basis S, then $\operatorname{rank}(G) = |S|$.

In particular:

- All bases of a free group G have the same cardinality
- Two free groups are isomorphic $\Leftrightarrow$ they have bases of same cardinality.

COVERING MAPS BETWEEN GRAPHS

<u>Def</u> $f : X \to Y$ morphism of graphs is a <u>covering map</u> if it is a local isomorphism: $\forall x \in X^0$ $f_x : st_X(x) \to st_Y(f(x))$ is a bijection.

<u>Exercise</u> $f : X \to Y$ morphism. It is a covering map $\iff$ the induced map between realizations is a covering of topological spaces.

A covering automorphism (or deck transformation) is $\varphi \in \mathrm{Aut}(X)$ such that
$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$
commutes.

<u>Observation</u> Topological deck transf. $\leftrightarrow$ graph deck transf.

<u>Obs.</u> φ has no inversions. $\left(\varphi(e) = \bar{e} \Rightarrow f(e) = f(\varphi(e)) = \right.$
$\left. = f(\bar{e}) = \overline{f(e)} \ \lightning \right)$

THE TREE OF REDUCED PATHS

Let X be a connected graph, $x_0 \in X^0$.

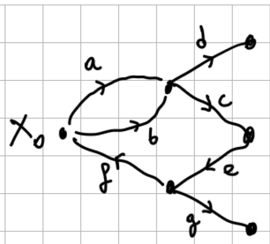
For $n = 1, 2, \ldots$ define $V_n = \{$ reduced paths in X with initial vertex x_0, of length $n \}$

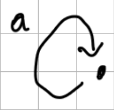
We have a sequence of functions

$\{x_0\} \leftarrow V_1 \leftarrow V_2 \leftarrow V_3 \leftarrow \ldots$

"forgetting the last step". This defines a tree T_{X, x_0}

Example





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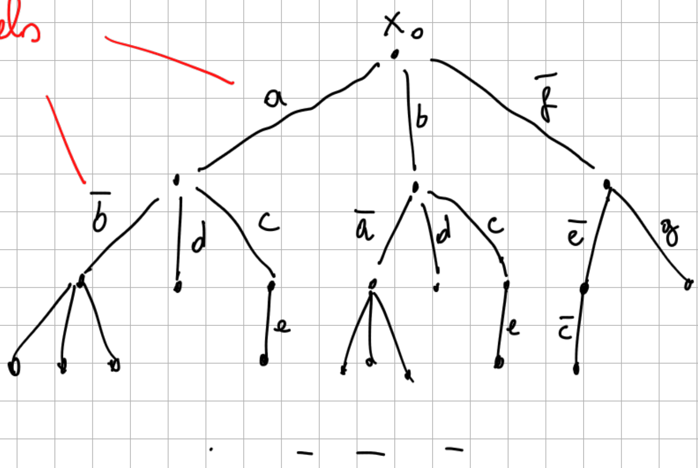
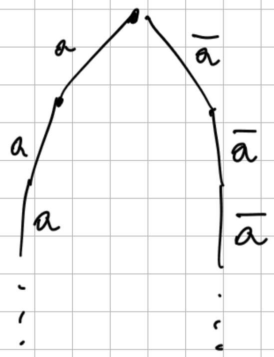


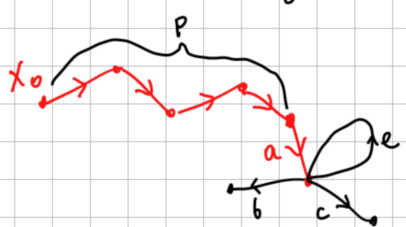


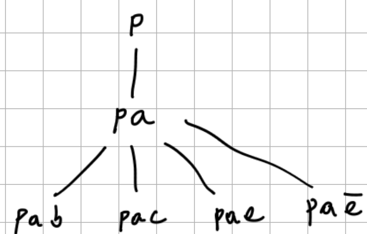
We have a morphism $(T_{X,x_0}, x_0) \to (X, x_0)$

$p \in T^\circ_{X,x_0} \longmapsto$ terminal vert. of p

(edges mapped in the "obvious" way)

obs It is a covering map.





Note: real(T_{X,x_0}) is contractible; in particular, it is simply connected. Therefore, real$(T_{X,x_0}) \to$ real(X) is a universal covering map.

Consequences: vertices projecting to x_0 = π_1 of the top. realization !!

- {closed reduced paths at x_0} $\longleftrightarrow \pi_1(X, x_0)$.

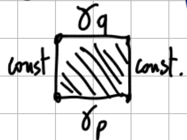
  $p \longmapsto [\gamma_p]$

  If p is a path in X, we denote by $\gamma_p : [0,1] \to \mathrm{real}(X)$ the induced path in the top. realization.

- If p is a closed reduced path at x_0, and $[\gamma_p] = 1 \in \pi_1(X, x_0)$, then p has length 0.

- In particular, a connected graph is simply connected $\iff$ it is a tree
  (from one of the exercises on the VIRTUALE page)

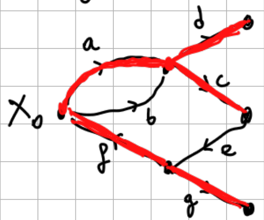
- $\forall$ path p in X $\exists!$ reduced path q with same endpoints s.t. γ_p and γ_p are homotopic relative to endpoints
  red$(p) := q$.



(if $e_1 \dots e_k$ and $e'_1, \dots, e'_\ell$ are reduced and homotopic, then $e_1 \dots e_k \bar{e}'_\ell \dots \bar{e}'_1$ "simplifies" completely $\Rightarrow$ they are equal)

<u>Prop</u> Let X be a connected graph, $x_0 \in X^0$. Then, $\pi_1(X, x_0)$ is free.

Pf: Fix a spanning tree $L \subseteq X$, and an orientation for edges outside L: $X^1 \setminus L^1 = E \sqcup \bar{E}$



$E = \{b, e\}$

$\bar{E} = \{\bar{b}, \bar{e}\}$

F_E free group with basis E.

Notation: if $\bar{e} \in \bar{E}$, we consider it as an element of F_E:

$$\bar{e} = e^{-1}.$$

For $e \in E \cup \bar{E}$, define $p(e) = [x_0, \alpha(e)]_L \, e \, [\omega(e), x_0]$. (concatenation)

Then, consider the homomorphism

$$F_E \longrightarrow \pi_1(X, x_0)$$
$$e \in E \longmapsto [\gamma_{p(e)}].$$

ESAMPLES
$p(b) = b\bar{a}$
$p(e) = a c e f$
$p(\bar{e}) = \bar{f}\bar{e}\bar{c}\bar{a}$
(graph above)

It sends a generic element of F_E

$$e_1 \dots e_k \longmapsto [\gamma_{p(e_1)}] \dots [\gamma_{p(e_k)}] = [\gamma_{p(e_1) \dots p(e_k)}]$$

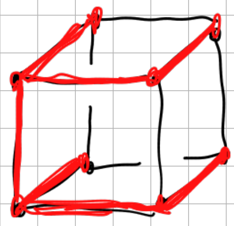
$\boxed{e_i \in E \cup \bar{E},\ e_{i+1} \neq \bar{e}_i.}$

$[\gamma_{p(\bar{e})}] = [\gamma_{p(e)}]^{-1}$

$$\mathrm{red}(p(e_1) \dots p(e_k)) = [x_0, \alpha(e_1)]_L \, e_1 \underbrace{[\omega(e_1), \alpha(e_2)]_L}_{\substack{\text{reduction of} \\ [\omega(e_1), x_0]_L [x_0, \alpha(e_2)]_L}} e_2 \dots [\omega(e_k), x_0]_L .$$

Any closed reduced path $/x_0$ is of that form (in a unique way), so,
$F_E \to \pi_1(X, x_0)$ is a bijection.

__Example__

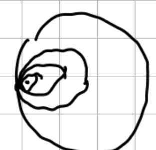


The fundamental group is free of rank 5

In general, if X is finite, any sp. tree has $|X^0| - 1$ edges

$$\Rightarrow \operatorname{rank}(\pi_1(X, x_0)) = \frac{1}{2}|X^1| - |X^0| + 1$$

__Non-example__ (Hawaiian earring)



The fundamental group is __not__ free.

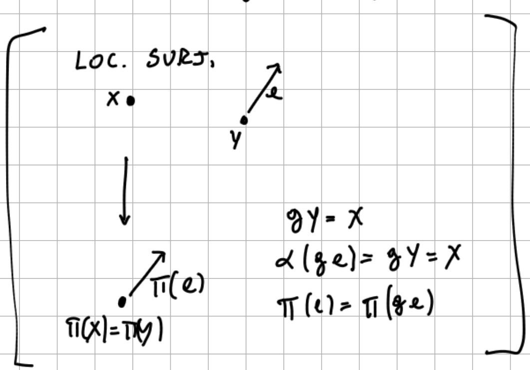
FREE ACTIONS

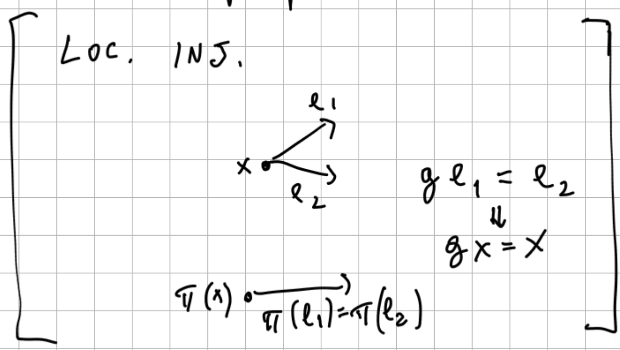
__Def__ Let Y be a graph, G be a group.
An action $G \curvearrowright Y$ is __free__ if

- there is no inversion: $ge \neq \bar{e} \quad \forall e \in Y^1$
- $\forall v \in Y^0 \quad \forall g \in G - \{1\} \quad gv \neq v$

NOTE: G acts on the top. realization with homeomorphisms. The action is free $\Leftrightarrow$ the top. action is free in the usual sense.

__obs__ $G \curvearrowright Y$ free $\Rightarrow \pi : Y \to G \backslash Y$ is a covering map, and, if Y is connected, $G \cong$ group of deck transformations $< \operatorname{Aut}(Y)$.
(local surjectivity holds in general when quotienting by a group; local injectivity is a consequence of freeness).

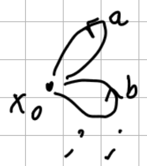




__Example__ X connected, $x_0 \in X^0$. $\pi_1(X, x_0) \curvearrowright T_{X, x_0}$ freely, as $\pi_1(X, x_0) \cong \{\text{deck transformation}\}$. The action is:

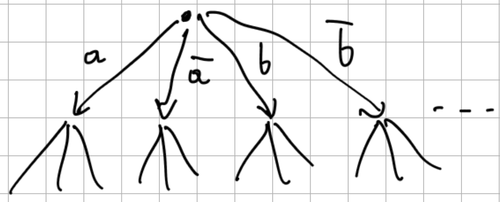
$$[\gamma_p] \cdot q = \operatorname{red}(pq) \quad \text{(on vertices)}$$

In particular: X bouquet



any cardinality of edges.

T_{X,x_0}:



Fix orientation $X^1 = E \sqcup \overline{E}$, $E = \{a, b, \dots\}$

Then $\{\text{vertices of } T_{X,x_0}\} \longleftrightarrow F_E$

The action $\pi_1(X, x_0) \cong F_E \curvearrowright T_{X,x_0}$, on vertices, is the multiplication in F_E: $g \cdot h = gh$

Theorem A group F is free $\Leftrightarrow$ it acts freely on a tree.

Pf: $\Rightarrow$ by the example above (if S is a basis of F, take a bouquet with $|S|$ geometric edges, so that $E \leftrightarrow S$ and $F \cong F_E \curvearrowright T_{X,x_0}$)

$\Leftarrow$: T tree, $F \curvearrowright T$ free action.

The quotient $X = F \backslash T$ is a connected graph.

$F \underset{\uparrow}{\cong}$ covering automorphisms $\underset{\uparrow}{\cong} \pi_1(X, x_0)$, which is free.

T connected ... universal cover

$\square$

Theorem (Shreier's theorem)

Let F be a free group. Every subgroup of F is free.

Pf: F acts freely on a tree T.

If $H < F$ is a subgroup, then $H \curvearrowright T$ by composition:

$$H \hookrightarrow F \longrightarrow \mathrm{Aut}(T),$$

and the action is free.

More explicitly, we can take $T = T_{X,x_0}$ the universal cover of a bouquet with $\{\text{geometric edges}\} \leftrightarrow S$ a basis of F

$\{\text{vertices of } T\} \leftrightarrow F$, on which $F \cong \pi_1(X, x_0)$, and also $H < F$, act by multiplication. Then, $H \cong \pi_1(H\backslash T)$. Describe $H\backslash T$:

$\{\text{vertices of } H\backslash T\} \leftrightarrow H\backslash F = \{Hx : x \in F\}$

cosets

$(H\backslash T)^1$? $\forall x \in F$ $\forall s \in S$ we have

$x \longrightarrow xs$ edge of T (and its inverse) $\rightsquigarrow$ $Hx \longrightarrow Hxs$ edge of $H\backslash T$ (and its inverse)

$s \in S$

might be the same vertex
(Condition: $xsx^{-1} \in H$)

NOTE: $st(x) \leftrightarrow S \sqcup \overline{S}$ $\qquad xs^{-1} \longleftarrow x \longrightarrow xs$

$st(Hx)$ $\swarrow\!\!\nearrow$

a set of "formal inverses"

Let G be a group, and fix a generating system:
a function $\underset{\text{set}}{S} \longrightarrow G$ whose image generates G.

Then, we have a surjective hom. $F_S \twoheadrightarrow G$.
$H := \text{Kernel} < F_S$.
$H \cong \pi_1(H \backslash T)$ as above.

$$(H\backslash T)^0 \longleftrightarrow H \backslash F_S \cong G.$$

$(H\backslash T)^1$: $\forall g \in G$, $\forall s \in S$, an (oriented) edge $\overset{g}{\bullet} \longrightarrow \overset{gs}{\bullet}$ (g, s).

and an inverse $\overset{g}{\bullet} \longleftarrow \overset{gs}{\bullet}$ $(gs, \bar{s})$.

NOTE: $$\mathrm{st}_{H\backslash T}(g) = \{(g,s) : s \in S\} \sqcup \{(gs, \bar{s}) : s \in S\}$$
$$\subseteq G \times (S \sqcup \bar{S}) \hookleftarrow (H\backslash T)^1.$$

NOTE: $H\backslash T$ is isomorphic to the CAYLEY GRAPH of G w.r.t. the generating system S.

<u>Note</u>: We can "read" from it a basis of the Kernel of $F_S \twoheadrightarrow G$.