Recall what we did last time:

- Given a graph of groups (Y, G), we defined its FUNDAMENTAL GROUP $\pi_1(Y, G; T)$ or $\pi_1(Y, G; P_0)$, relative to a "base" spanning tree T or vertex P_0, whose choice gives the same result up to isomorphism,

- We defined words over (Y, G). Each word has a type which is a path in the graph Y. To any word we associate an element of a group $F(Y, G)$ that we defined (which has $\pi_1(Y, G; T)$ as a quotient, i.e., we have a surjective hom.
$$F(Y, G) \twoheadrightarrow \pi_1(Y, G; T).\Big)$$
Any word can be "simplified", without changing the associated element of $F(Y, G)$, until it becomes REDUCED.

- We proved a technical proposition, which says: the element $\in F(Y, G)$ associated to a nontrivial, reduced word is $\neq 1 \in F(Y, G)$.
This has important consequences. In particular:

Fix (from now on) a spanning tree $T \subseteq Y$, and denote, for simplicity, $\pi := \pi_1(Y, G; T)$.
A corollary of the technical proposition is that the natural maps
$$G_P \longrightarrow \pi \quad ,$$
where $P \in Y^0$, are all injective. Henceforth, we identify the vertex groups G_P with their image in π, so they become subgroups of π.

What about edge groups G_y, for $y \in Y$?
We could see G_y as a subgroup of $G_{\alpha(y)}$ or of $G_{\omega(y)}$, which now are different subgroups of π.
In order to make and fix these choices, we
<u>fix an orientation</u> : $Y^1 = Y^1_+ \sqcup Y^1_-$.

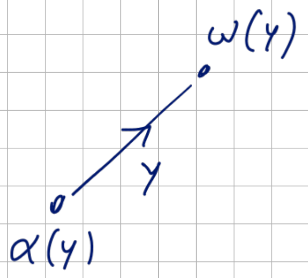
Then, for $y \in Y^1_+$, we view $G_y < G_{\alpha(y)} < \pi$, so that also edge groups are, from now on, identified with subgroups of π.

NOTE: if $y \in Y^1_-$, then $G_y = G_{\bar{y}} < G_{\alpha(\bar{y})} = G_{\omega(y)} < \pi$

NOTE: the choice does <u>not</u> matter for "tree edges": if $y \in T^1$, then

$$\begin{tikzcd} & G_{\alpha(y)} \arrow[dr, hook] & \\ G_y \arrow[ur, hook] \arrow[dr, hook] & & \pi \\ & G_{\omega(y)} \arrow[ur, hook] & \end{tikzcd}$$

commutes, because:
for $a \in G_y$, we have $y\, a^y\, \bar{y} = a^{\bar{y}}$ in $F(Y, G)$
$\Rightarrow$ $a^y = a^{\bar{y}}$ in π (if $y \in T^1$)



COMMENT: up to now, "fundamental groups of graph of groups" may be considered just as an interesting (?) construction of groups, "built" from "simpler groups" which are the vertex (and edge) groups, and we know something about their structure (we can write presentations, write and manipulate elements using words ...). In a few pages, we will see

Now this is useful because, whenever we have a group acting on a tree, then it acquires such structure!

But for now we do the opposite:
given the data above (Y, G, the orientation on Y^1)

- a graph $\widetilde{X}$ with an action $\pi \curvearrowright \widetilde{X}$ (without inversions)
- a morphism $f : \widetilde{X} \to Y$

such that

- f is π-invariant, inducing $\pi \backslash \widetilde{X} \xrightarrow{\cong} Y$ (iso.)
- we have "lifts" $v \in Y^0 \rightsquigarrow \tilde{v} \in \widetilde{X}^0$ with $\mathrm{Stab}_\pi(\tilde{v}) = G_v$

$y \in Y^1 \rightsquigarrow \tilde{y} \in \widetilde{X}^1$ with $\mathrm{Stab}_\pi(\tilde{y}) = G_y$

these are honest equalities!
Keep in mind that vertex and edges groups G_v, G_y are considered as subgroups of π.

CONSTRUCTION:

$$\widetilde{X}^0 \overset{\text{def.}}{=} \bigsqcup_{v \in Y^0} \pi / G_v, \qquad f(\gamma G_v) \overset{\text{def.}}{=} v \qquad \tilde{v} \overset{\text{def.}}{=} 1\, G_v$$

$$\widetilde{X}^1_+ \overset{\text{def.}}{=} \bigsqcup_{y \in Y^1_+} \pi / G_y, \qquad f(\gamma G_y) \overset{\text{def.}}{=} y \qquad \tilde{y} \overset{\text{def.}}{=} 1\, G_y$$

$$\alpha(\gamma G_y) \overset{\text{def.}}{=} \gamma\, G_{\alpha(y)} \qquad (\text{ok because } G_y < G_{\alpha(y)})$$
$$\omega(\gamma G_y) \overset{\text{def.}}{=} \gamma\, y\, G_{\omega(y)} \qquad (\text{exercise: check this def. is well posed}).$$

This is an oriented graph; for the actual graph, we add the opposite edges as usual:

$$\begin{bmatrix} \alpha(\overline{\gamma \tilde{y}}) = \omega(\gamma \tilde{y}) = \gamma y\, \widetilde{\omega(y)} = \gamma y\, \widetilde{\alpha(\overline{y})} \\ \omega(\overline{\gamma \tilde{y}}) = \alpha(\gamma \tilde{y}) = \gamma\, \widetilde{\alpha(y)} = \gamma\, \widetilde{\omega(\overline{y})}. \end{bmatrix}$$

NOTE: T lifts to $\widetilde{T} \subset \widetilde{X}$ (using the lifts $v \to \tilde{v}$, $y \to \tilde{y}$ above)

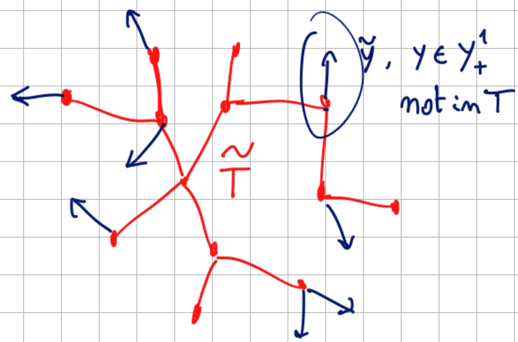
because, if $y \in T^1$, then $y = 1 \in \pi$.

Theorem The $\widetilde{X}$ defined above is a tree. (ANOTHER TECHNICAL PROOF, SORRY! STATEMENT IS MORE IMPORTANT THAN PROOF)

Proof: First, we prove that $\widetilde{X}$ is connected.

Essentially, this is true because π is generated by the union of the G_v's and y for $y \in Y_+^1 \setminus T^1$:

consider $W \subset \widetilde{X}$ subgraph consisting of $\widetilde{T}$ and of the lifts $\widetilde{y}$ of the positive edges $y \in Y_+^1 \setminus T^1$



If $\gamma \in G_v$, then W and γW both contain the vertex $\widetilde{v}$; therefore, $W \cup \gamma W$ is connected.
If $y \in Y_+^1 - T^1$, then W and yW both contain the vertex $\omega(\widetilde{y}) = \omega(G_y) = y\, G_{w(y)}$.
These observations imply that $\widetilde{X}$ is connected.

<u>It remains to show</u>: there is no locally injective closed path of length $n > 0$.

Take such a path $\gamma_1 \widetilde{y}_1^{\varepsilon_1}, \dots, \gamma_m \widetilde{y}_m^{\varepsilon_m}$ $\qquad \varepsilon_i \in \{1,-1\}$.
$\qquad \delta_i \in \{0,1\}$.

$\alpha(\gamma_1 \widetilde{y}_1^{\varepsilon_1}) = \gamma_1 y_1^{\delta_1} \widetilde{\alpha(y_1^{\varepsilon_1})}$ (by def. of $\widetilde{X}$)

$\omega(\gamma_1 \widetilde{y}_1^{\varepsilon_1}) \overset{\text{def. of } \widetilde{X}}{=} \underline{\gamma_1 y_1^{1-\delta_1} \widetilde{\omega(y_1^{\varepsilon_1})}} \overset{\text{consecutive edges}}{=} \underline{\gamma_2 y_2^{\delta_2} \widetilde{\alpha(y_2^{\varepsilon_2})}} \overset{\text{def.}}{=} \alpha(\gamma_2 \widetilde{y}_2^{\varepsilon_2})$

$\vdots$

$\omega(y_1^{\varepsilon_1}) = \alpha(y_2^{\varepsilon_2}) =: V_1$

$\gamma_2 y_2^{\delta_2} = \gamma_1 y_1^{1-\delta_1} g_1$, for some $g_1 \in G_{V_1}$.

Define $V_0, V_1, V_2, \dots$

In the same way, define $V_2, g_2, V_3, g_3, \dots$ and we have:

$$\gamma_3 \, y_3^{\delta_3} = \gamma_2 \, y_2^{1-\delta_2} g_2 \overset{\text{def. of } \delta_2}{=} \gamma_2 \, y_2^{\delta_2} \, y_2^{\varepsilon_2} g_2 \overset{\text{previous equality}}{=} \gamma_1 \, y_1^{1-\delta_1} g_1 \, y_2^{\varepsilon_2} g_2$$

$$\vdots$$

$$\gamma_n \, y_n^{\delta_n} = \gamma_1 \, y_1^{1-\delta_1} g_1 \, y_2^{\varepsilon_2} g_2 \, y_3^{\varepsilon_3} \dots y_{n-1}^{\varepsilon_{n-1}} g_{n-1}$$

We also have: $\quad \alpha(y_1^{\varepsilon_1}) = w(y_n^{\varepsilon_n}) =: v_0$

("closing the circle") $\quad \gamma_1 \, y_1^{\delta_1} = \gamma_n \, y_n^{1-\delta_n} g_0 \, , \quad g_0 \in G_{v_0}$

$$\Rightarrow \quad \gamma_1 \, y_1^{\delta_1} = \gamma_1 \, y_1^{1-\delta_1} g_1 \, y_2^{\varepsilon_2} \dots g_{n-1} \, y_n^{\varepsilon_n} g_0$$

$$1 = y_1^{\varepsilon_1} g_1 \, y_2^{\varepsilon_2} \dots y_n^{\varepsilon_n} g_0$$

$g_i \in G_{v_i}$

$$\boxed{1 = g_0 \, y_1^{\varepsilon_1} g_1 \, y_2^{\varepsilon_2} \dots y_n^{\varepsilon_n} \in \pi}$$

$c = (y_1^{\varepsilon_1}, y_2^{\varepsilon_2} \dots y_n^{\varepsilon_n})$ closed path in Y

$\mu = (g_0, g_1, \dots, g_{n-1}, 1)$ CANNOT BE REDUCED, since the associated element is trivial (use the previous proposition)

$\rightsquigarrow y_i = y_{i+1}, \; -\varepsilon_i = \varepsilon_{i+1}, \; g_i = 1$ for some $i \in \{1, \dots, n-1\}$.

$\Rightarrow \gamma_i = \gamma_{i+1} \rightsquigarrow$ the initial path in $\tilde{X}$ is <u>not</u> loc. inj. $\square$

THE CONCLUSION TO KEEP IN MIND IS THAT π ACTS NATURALLY ON A CERTAIN TREE, WITH VERTEX AND EDGE STABILIZERS WHICH ARE EXACTLY (CONJUGATES OF) VERTEX AND EDGE GROUPS, AND WITH QUOTIENT Y.

The Bass-Serre tree.

Now, we do the opposite: start from an action (without inversions) $\pi \curvearrowright X$ on a connected graph X. Let $Y = \pi \backslash X$.

Any group

Fix a spanning tree $T \subset Y$, and a lift $\tilde{T} \subset X$.

Also, fix an orientation $Y^1_+ \subset Y^1$.

For $y \in Y^1_+$, we fix $\tilde{y}$ a lift of y with $\alpha(\tilde{y}) \in \tilde{T}$,

and $\gamma_y \in \pi$ such that $w(\tilde{y}) \in \gamma_y \tilde{T}$, with $\gamma_y = 1$ if $y \in T^1$.

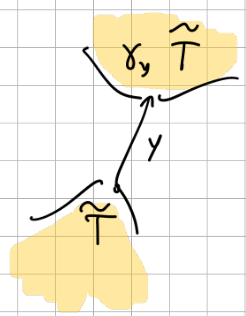
Then, consider the graph of groups (Y, G) with:

$$G_p = \mathrm{Stab}_\pi(\tilde{p}) \,;$$

$$G_y = \mathrm{Stab}_\pi(\tilde{y}) \quad \text{for } y \in Y^1_+ \,, \quad \text{with}$$

$$G_y \hookrightarrow G_{\alpha(y)} \quad \text{the inclusion} \quad \left(\text{note: } \alpha(\tilde{y}) = \widetilde{\alpha(y)}\right)$$

$$G_y \hookrightarrow G_{w(y)}$$
$$g \longmapsto \gamma_y^{-1} g \gamma_y$$



<u>OBS</u> We have a homomorphism

$$\pi_1(Y, G; T) \xrightarrow[\varphi]{} \pi.$$

$(G_p \to \pi$ by inclusion; $\quad y \to \gamma_y)$

check: relations $\mapsto 1 \in \pi$

NOW, WE PROVE THAT A GROUP ACTING ON A TREE IS ISOMORPHIC TO THE FUNDAMENTAL GROUP OF THE "QUOTIENT GRAPH OF GROUPS". THUS, IF WE KNOW VERTEX AND EDGE STABILIZERS, WE UNDERSTAND (TO A CERTAIN EXTENT) THE STRUCTURE OF THE ACTING GROUP.

$\boxed{\text{Thm}}$ If X is a tree, then $\pi_1(Y, G; T) \xrightarrow{\cong} \pi$.

Pf: Let $\tilde{X}$ be the Bass-Serre tree built from the graph of groups.

Define $\psi : \tilde{X} \to X$ sending
$$\psi(\gamma \tilde{v}) = \varphi(\gamma) \cdot \tilde{v},$$
$$\psi(\gamma \tilde{y}) = \varphi(\gamma) \cdot \tilde{y} \qquad y \in Y^1_+ .$$

same notation for different things!

Is ψ a graph morphism? We check the conditions:

$$\alpha(\psi(\gamma\tilde{y})) = \alpha(\varphi(\gamma)\cdot\tilde{y}) = \varphi(\gamma)\cdot\alpha(\tilde{y})$$
$$\psi(\alpha(\gamma\tilde{y})) = \psi(\gamma\,\alpha(\tilde{y})) = \psi(\gamma\,\widetilde{\alpha(y)}) = \varphi(\gamma)\cdot\widetilde{\alpha(y)} \quad \checkmark$$
α is respected

$$\omega(\psi(\gamma\tilde{y})) = \omega(\varphi(\gamma)\cdot\tilde{y}) = \varphi(\gamma)\cdot\omega(\tilde{y}) = \varphi(\gamma)\,\gamma_y\,\widetilde{\omega(y)}$$
$$\psi(\omega(\gamma\tilde{y})) = \psi(\gamma\,\omega(\tilde{y})) = \psi(\gamma\, y\,\widetilde{\omega(y)}) = \varphi(\gamma)\,\gamma_y\,\widetilde{\omega(y)} \quad \checkmark$$
ω is respected

By construction, $\psi : \tilde{X} \to X$ is φ-equivariant:
$$\psi(\gamma u) = \varphi(\gamma)\,\psi(u) \quad \text{for } u \text{ vertex or edge.}$$

Claim: ψ is surjective.

$W :=$ subgraph of X spanned by the $\tilde{y}$, for $y \in Y^1_+$.

Note that $\pi W = X$, and $\psi(\tilde{X})$ contains W.

Then, $\varphi : \pi_1(Y, G; T) \to \pi$ is surjective because:

If, for some $g \in \pi$, $W \cap gW \neq \emptyset$,
then they share a vertex $\tilde{v} \in W \cap gW$, with $v \in Y^0$;
or a vertex $\gamma_y \tilde{v}$, with $y \in Y^1_+$, $v \in Y^0$.
It follows that π is generated by the $\varphi(G_v)$, $\varphi(y)$.

Since φ is surjective, also ψ is surjective.

Claim: Ψ is locally injective.

By equivariance, it is enough to check this at vertices $\tilde{v} \in \tilde{X}^0$, for $v \in Y^0$.

$\mathrm{Star}(G_v)$ contains: • γG_y with y positive, $\alpha(y) = v$, $\gamma \in G_v$

• $\overline{\gamma G_y}$ with y positive, $\omega(y) = v$, $\gamma y \in G_v$

they are sent respectively to: $\mapsto \varphi(\gamma)\tilde{y}$

$\mapsto \overline{\varphi(\gamma)\tilde{y}}$

If $\varphi(\gamma_1)\tilde{y} = \varphi(\gamma_2)\tilde{y}$, then $\varphi(\gamma_1^{-1}\gamma_2) \in G_y$, so $\gamma_1^{-1}\gamma_2 \in G_y$ (φ is "identity" on G_v) ...

Idem...

We have shown that Ψ is surjective and locally injective.

NOTE: we haven't used that X is a tree.

In fact, we now deduce that the following conditions are equivalent (as for amalgamated product, which correspond to the particular case $Y = \bullet\!\!\longrightarrow\!\!\bullet$):

(1) X is a tree

(2) $\Psi : \tilde{X} \to X$ is a isomorphism

(3) $\varphi : \pi_1(Y, G; T) \longrightarrow \pi$ is a isomorphism.

In fact:

- If X is a tree, then we use the general fact that a locally injective morphism from a connected graph to a tree is injective, and obtain (2).
- If $\Psi : \tilde{X} \to X$ is a isomorphism, suppose that $\gamma \in \operatorname{Ker}\varphi$, with $\gamma \neq 1$. Then $\gamma \notin G_v$ (of any v), so G_v and γG_v are distinct vertices of $\tilde{X}$, with the same image (v) in X, contradiction! $\Rightarrow$ (3) holds.

EXERCISE: prove the remaining implication.

In any case, the statement of the theorem, which is the most important implication, is proved. $\square$