Simplicial spheres, maps between them, and the simplicial volume of Davis' manifolds

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A partial order for triangulated spheres

Let S and T be simplicial complexes homeomorphic to the n-sphere S^n, for some $n \geq 1$.
S dominates T (notation: $T \leq S$) if there is a simplicial map $f : S \to T$ with $\deg(f) \neq 0$.

For every n we obtain a poset

$$\left(\left\{ \begin{matrix} \text{triangulations of } S^n \\ \text{up to isomorphism} \end{matrix} \right\}, \leq \right).$$

Example: Triangulations of the circle.

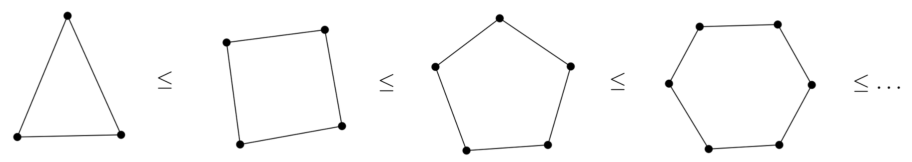


Figure 1. The poset for $n = 1$. We actually get a linear order.

For $n \geq 2$ the poset is much more complicated.

I would like to understand the structure of some specific subposets of the ones just defined. The motivation comes from the study of an invariant of manifolds: the simplicial volume.

A specific subposet that I care about

Consider triangulations of the **3-sphere**. But not all of them: only the ones that are **flag**.

What is a *flag* simplicial complex?
S is flag if every subset of pairwise-adjacent vertices spans a simplex in S.
In other words, S is the maximal simplicial complex with a given 1-skeleton.

Consider a further subposet, given by **flag 3-spheres with nonzero γ_2**.

If S is a triangulation of S^3 (there is a more general definition), then

$$\gamma_2(S) = 16 - 8v(S) + 4e(S) - 2f(S) + t(S) = \mathbf{16 - 5v(S) + e(S)},$$

where v, e, f, t denote the number of $0, 1, 2, 3$-simplices.

Theorem (Davis, Okun): $\boldsymbol{\gamma_2(S) \geq 0}$ if S is a flag 3-sphere.

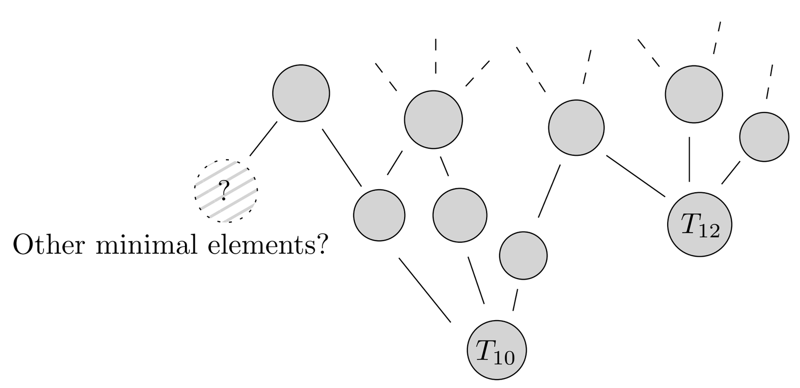
Problem: Find the **minimal elements** of the poset

$$\left(\left\{ \begin{matrix} \text{flag triangulations of } S^3 \text{ with } \gamma_2 > 0, \\ \text{up to isomorphism} \end{matrix} \right\}, \leq \right).$$

Is there a finite number of them, or are they infinite?

What I have found up to now.

There are at least two distinct minimal elements:



- T_{10} has 10 vertices, and is the join of two pentagons;
- T_{12} is a triangulation with 12 vertices, already described in a preprint by L. Venturello.

With a computer, **I have generated thousands** of flag 3-spheres with $\gamma_2 > 0$; all of them dominate T_{10} or T_{12}. Checking this is not that easy, I had to invent a sufficiently effective algorithm.

Motivation, Part I — Simplicial volume

The simplicial volume is a **numerical homotopy invariant** for compact topological manifolds.

$$M \text{ manifold} \rightsquigarrow \|M\| \in \mathbb{R}_{\geq 0}$$

The simplicial volume $\|M\|$ is a **nonnegative real number**. Usually, we aren't interested in the precise value, but in whether it is **zero or positive**. Some typical examples:

- If M is a Riemannian manifold with strictly **negative sectional curvature**, then $\|M\| > 0$;
- If M is a Riemannian manifold with **nonnegative sectional curvature**, then $\|M\| = 0$.

In general, it is uncomputable. An algorithm cannot accept (triangulated) manifolds M in input and decide whether $\|M\| = 0$ or $\|M\| > 0$.

Gromov conjectured a relation between the simplicial volume and the Euler characteristic of **aspherical** manifolds.

The question of Gromov: Does the implication

$$\|M\| = 0 \Longrightarrow \chi(M) = 0$$

hold for closed aspherical manifolds?

This has become a central question in the community studying simplicial volume. My idea is to test it in a particular class of manifolds, arising from a construction of Michael W. Davis.

Motivation, Part II — Davis' manifolds

Davis' construction as a black box.

Input	Output
A simplicial complex T	A topological space $M(T)$

Actually, $M(T)$ has more structure: it is a cube complex. Some important properties:

If ...	then ...
T is homeomorphic to S^n ...	$M(T)$ is a $(n+1)$-manifold.
T is flag ...	$M(T)$ is aspherical.

In particular:

flag triangulation of a sphere $\rightsquigarrow$ aspherical manifold.

My idea is to test Gromov's conjecture on the manifolds obtained in this way. However, understanding whether $\|M(S)\|$ is positive or vanishes seems an hard task. After some work, I proved the following result.

Let S and T be triangulations of S^n. **If $T \leq S$, then $\|M(T)\| \leq \|M(S)\|$.**

The case $n=2$

For flag triangulations of S^2, I have found the following characterization:

Let S be a flag triangulation of S^2.
$\|M(S)\| > 0$ if and only if $S \geq T_9$.

T_9 is the flag triangulation of S^2 in Figure 2.
In other words, the poset

$$\left(\left\{ \begin{matrix} \text{flag triangulations of } S^2 \text{ giving} \\ \text{positive simplicial volume,} \\ \text{up to isomorphism} \end{matrix} \right\}, \leq \right).$$

has only one minimal element: T_9.

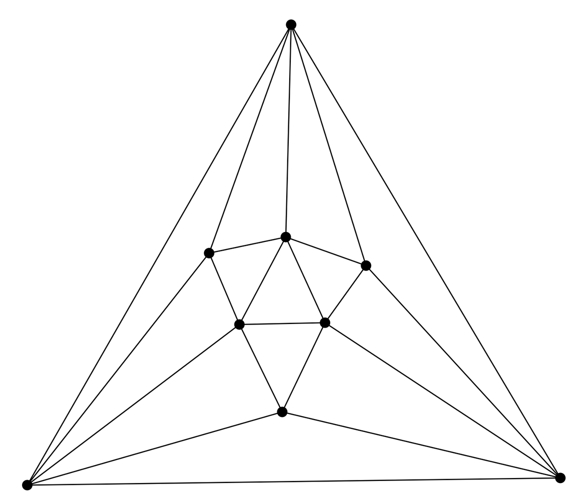


Figure 2. The 1-skeleton of the flag 2-sphere T_9.

Graph minors

For $n = 2$, I have also found a connection with the theory of graph minors.

A **minor of a graph G** is a graph obtained from G with a sequence of three types of operations:

- Erasing an edge;
- Erasing a vertex;
- Collapsing an edge.

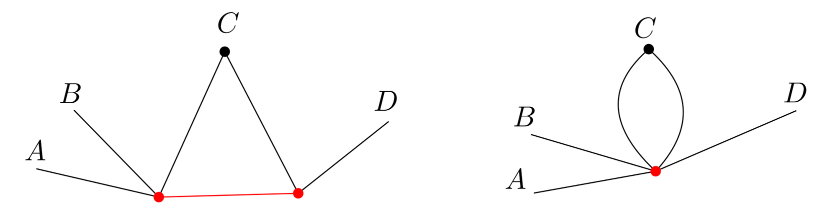


Figure 3. The collapse of the red edge.

Let S and T be triangulations of S^2. **If $T^{(1)}$ is a minor of $S^{(1)}$, then $T \leq S$.**
More precisely, there is a simplicial map $f : S \to T$ with $|\deg(f)| = 1$.

Can we do something similar for triangulations of higher-dimensional spheres?

Graph minor theorems and their consequences

Robertson and Seymour, in a long series of papers, established very deep results about the notion of graph minors. This is among the most important ones:

Graph minor theorem. Let $G_1, G_2, \ldots$ be an infinite sequence of (finite) graphs. Then there are indices $i < j$ such that G_i is a minor of G_j.

In short, the minor relation is a **well-quasiorder** on finite graphs. They also proved:

Let H be a graph. There is a **polynomial-time algorithm** that given a graph G decides whether H is a minor of G.
However, the same algorithmic problem with H not fixed is NP-complete.

From the connection with the relation $\leq$ for 2-spheres, we deduce interesting consequences:

- Triangulations of S^2 are well-quasiordered by $\leq$. In particular, **every subposet has a finite number of minimal elements** up to isomorphism;
- For every $T \cong S^2$, there is $d_T \in \mathbb{N}$ such that
  $$S \geq T \implies \exists f : S \to T \text{ with } 0 < |\deg(f)| \leq d_T;$$
- For every $T \cong S^2$, there is a **polynomial-time algorithm** to decide (given S) whether $S \geq T$.

- Can we always take $\boldsymbol{d_T = 1}$?
- Is there a **universal** polynomial-time algorithm that works for every T?
- Can we extend these results for triangulations of **S^n, for some $n > 2$**?

The case $n=3$

This is the lowest dimension in which we can really test Gromov's question. The Euler characteristic $\chi(M(S))$, for $S \cong S^3$, is easily computed: $\boldsymbol{\chi(M(S)) = 2^{v(S)-4} \cdot \gamma_2(S)}$.

Two steps to check Gromov's conjecture:

- Find the minimal elements of the subposet of flag 3-spheres S with $\gamma_2(S) > 0$;
- Check if $\|M(S)\| > 0$ for S such a minimal element.

I know that $\|M(T_{10})\| > 0$, but **I still don't know if $\|M(T_{12})\| > 0$.**

References

Ask me, or scan the QR code in the corner.